

**APPLIED MATHS I DEC 2019 PAPER SOLUTIONS**

**Q1)a) If  $\sin(\theta + i\phi) = \tan \alpha + i \sec \alpha$ , then show that  $\cos 2\theta \cdot \cosh 2\phi = 3$ . (5M)**

**Ans :** We have  $\sin(\theta + i\phi) = \tan \alpha + i \sec \alpha$

$$\therefore \sin \alpha \cos i\phi + \cos \theta \sin i\phi = \tan \alpha + i \sec \alpha$$

$$\therefore \sin \alpha \cosh \phi + i \cos \theta \sinh \phi = \tan \alpha + i \sec \alpha$$

Equating real and imaginary parts ,

$$\tan \alpha = \sin \theta \cosh \phi$$

$$\sec \alpha = \cos \theta \sinh \phi$$

But

$$\sec^2 \alpha - \tan^2 \alpha = 1$$

$$\therefore \cos^2 \theta \sinh^2 \phi - \sin^2 \theta \cosh^2 \phi = 1$$

$$\left( \frac{1 + \cos 2\theta}{2} \right) \left( \frac{\cosh 2\phi - 1}{2} \right) - \left( \frac{1 - \cos 2\theta}{2} \right) \left( \frac{1 + \cosh 2\phi}{2} \right) = 1$$

$$\therefore \cosh 2\phi - 1 + \cos 2\theta \cosh 2\phi - \cos 2\theta - 1 - \cosh 2\phi + \cos 2\theta + \cos 2\theta \cosh 2\phi = 4$$

$$\therefore 2 \cos 2\theta \cosh 2\phi = 6$$

$$\therefore \cos 2\theta \cosh 2\phi = 3$$

**Q1)b) If  $u = \log(\tan x + \tan y)$ , then show that  $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = 2$ . (5M)**

**Ans :** We have

$$\frac{\partial u}{\partial x} = \frac{1}{(\tan x + \tan y)} \sec^2 x$$

$$\therefore \sin 2x \frac{\partial u}{\partial x} = 2 \sin x \cos x \frac{1}{(\tan x + \tan y)} \sec^2 x = 2 \cdot \frac{\tan x}{\tan x + \tan y}$$

$$\text{Similarly, } \sin 2y \frac{\partial u}{\partial y} = 2 \cdot \frac{\tan y}{(\tan x + \tan y)}$$

$$\therefore \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = 2 \frac{\tan x + \tan y}{(\tan x + \tan y)} = 2.$$

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Similarly , prove that

$$\therefore \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

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Q1)c) Express the matrix  $A = \begin{bmatrix} 0 & 5 & -3 \\ 1 & 1 & 1 \\ 4 & 5 & 9 \end{bmatrix}$  as the sum of a symmetric and skew symmetric matrix.

(5M)

Ans : We have

$$A' = \begin{bmatrix} 0 & 1 & 4 \\ 5 & 1 & 5 \\ -3 & 1 & 9 \end{bmatrix}$$

$$\therefore A + A' = \begin{bmatrix} 0 & 6 & 1 \\ 6 & 2 & 6 \\ 1 & 6 & 18 \end{bmatrix}$$

$$A - A' = \begin{bmatrix} 0 & 4 & -7 \\ -4 & 0 & -4 \\ 7 & 4 & 0 \end{bmatrix}$$

Let  $P = \frac{1}{2}(A + A')$  and  $Q = \frac{1}{2}(A - A')$  .

But we know that P is symmetric and Q is skew-symmetric and  $A = P + Q$  .

$$\therefore A = P + Q = \begin{bmatrix} 0 & 3 & 1/2 \\ 3 & 1 & 3 \\ 1/2 & 3 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -7/2 \\ -2 & 0 & -2 \\ 7/2 & 2 & 0 \end{bmatrix}$$

The first matrix is symmetric and the second is skew-symmetric .

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**Q1)d) Expand  $\sqrt{1 + \sin x}$  in ascending powers of  $x$  upto  $x^4$  terms .**

**(5M)**

**Ans :** We have

$$\begin{aligned} \sqrt{1 + \sin x} &= \sqrt{\sin^2(x/2) + \cos^2(x/2) + 2 \sin(x/2) \cos(x/2)} \\ &= \sqrt{[\sin(x/2) + \cos(x/2)]^2} = \sin(x/2) + \cos(x/2) \\ &= \left(\frac{x}{2}\right) - \frac{1}{6}\left(\frac{x}{2}\right)^3 + \dots + 1 - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{1}{24}\left(\frac{x}{2}\right)^4 - \dots \\ &= \frac{x}{2} - \frac{x^3}{48} + \dots + 1 - \frac{x^2}{8} + \frac{x^4}{384} - \dots \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \dots \end{aligned}$$

**Q2)a) Find non-singular matrices P and Q such that PAQ is in normal form where,**

$A = \begin{bmatrix} 4 & 3 & 1 & 6 \\ 2 & 4 & 2 & 2 \\ 12 & 14 & 5 & 16 \end{bmatrix}$  . Also find the rank of A .

**(6M)**

**Ans :** We first write

$$A = I_3 A I_4$$

$$\begin{bmatrix} 4 & 3 & 1 & 6 \\ 2 & 4 & 2 & 2 \\ 12 & 14 & 5 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow \frac{R_1}{4},$$

$$\begin{bmatrix} 1 & 3 & 1 & 6 \\ 0.5 & 4 & 2 & 2 \\ 3 & 14 & 5 & 16 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 3 & 1 & 6 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$C_2 \rightarrow C_2 - 2C_1, C_4 - 4C_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{-C_2}{3}, \frac{-C_3}{2}, \frac{-C_5}{5}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 12/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & 3/2 & 4/5 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -1/5 \end{bmatrix}$$

$$C_3 - C_2, C_4 - C_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 12/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & 5/6 & -7/24 \\ 0 & -1/3 & 1/3 & 0 \\ 0 & 0 & 1/2 & -5/24 \\ 0 & 0 & 0 & -1/12 \end{bmatrix}$$

$$C_{34}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & -7/24 & 5/6 \\ 0 & -1/3 & 0 & 1/3 \\ 0 & 0 & -5/24 & 1/2 \\ 0 & 0 & -1/12 & 0 \end{bmatrix}$$

**Q2)b) If  $z = f(x, y)$  and  $x = u \cosh v, y = u \sinh v$ , prove that**

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2} \left(\frac{\partial z}{\partial v}\right)^2 \quad (6M)$$

**Ans :** We have ,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} \cosh v + \frac{\partial z}{\partial y} \sinh v \quad \text{----- (1)}$$

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And,  $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} u \sinh v + \frac{\partial z}{\partial y} u \cosh v$

$$\frac{1}{u} \cdot \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \sinh v + \frac{\partial z}{\partial y} \cosh v \quad \text{----- (2)}$$

Now squaring (1) and (2) and subtracting, we get

$$\left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2} \left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cosh^2 v + \left(\frac{\partial z}{\partial y}\right)^2 \sinh^2 v + 2 \left(\frac{\partial z}{\partial x}\right) \left(\frac{\partial z}{\partial y}\right) \cosh v \cdot \sinh v - \left(\frac{\partial z}{\partial x}\right)^2 \sinh^2 v - \left(\frac{\partial z}{\partial y}\right)^2 \cosh^2 v - 2 \left(\frac{\partial z}{\partial x}\right) \left(\frac{\partial z}{\partial y}\right) \cosh v \cdot \sinh v$$

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2} \left(\frac{\partial z}{\partial v}\right)^2$$

**Q2)c) Prove that**  $\log\left(\frac{(a-b)+i(a+b)}{(a+b)+i(a-b)}\right) = i\left(2n\pi + \tan^{-1} \frac{2ab}{a^2-b^2}\right)$ . **Hence evaluate**  $\log\left(\frac{1+5i}{5+i}\right)$ .

**(6M)**

**Ans :** Let  $a + b = A$ ,  $a - b = B$ .

$$\begin{aligned} \therefore \log\left[\frac{B+iA}{A+iB}\right] &= 2n\pi i + \log\left[\frac{B+iA}{A+iB}\right] \\ &= 2n\pi i + \log(B+iA) - \log(A+iB) \\ &= 2n\pi i + \left[\log\sqrt{B^2+A^2} + i \tan^{-1} \frac{A}{B}\right] - \left[\log\sqrt{A^2+B^2} + i \tan^{-1} \frac{B}{A}\right] \\ &= 2n\pi i + i \left[\tan^{-1} \frac{A}{B} - \tan^{-1} \frac{B}{A}\right] \end{aligned}$$

But  $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x-y}{1+xy}\right)$

$$\begin{aligned} \therefore \log\left[\frac{B+iA}{A+iB}\right] &= 2n\pi i + i \tan^{-1} \left[\frac{(A/B) - (B/A)}{1 + (A/B) * (B/A)}\right] \\ &= 2n\pi i + i \tan^{-1} \left(\frac{A^2 - B^2}{2AB}\right) \end{aligned}$$

But  $A^2 - B^2 = (a+b)^2 - (a-b)^2 = 4ab$  and  $AB = (a+b)(a-b) = a^2 - b^2$

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$$\therefore \log \left[ \frac{B+iA}{A+iB} \right] = 2n\pi i + i \tan^{-1} \frac{4ab}{2(a^2-b^2)} = i \left[ 2n\pi + \tan^{-1} \frac{2ab}{(a^2-b^2)} \right]$$

**Q3)a) If  $\alpha$  and  $\beta$  are the roots of the equation  $z^2 \sin^2 \theta - z \sin 2\theta + 1 = 0$ , then prove that  $\alpha^n + \beta^n = 2 \cos n\theta \operatorname{cosec}^n \theta$  and  $\alpha^n \beta^n = \cos ec^{2n} \theta$  .** (6M)

**Ans :** Solving the quadratic equation in  $z$  ,

$$z = \frac{\sin 2\theta \pm \sqrt{\sin^2 2\theta - 4 \sin^2 \theta}}{2 \sin^2 \theta} = \frac{2 \sin \theta \cos \theta \pm \sqrt{4 \sin^2 \theta \cos^2 \theta - 4 \sin^2 \theta}}{2 \sin^2 \theta}$$

$$\therefore z = \frac{\cos \theta \pm \sqrt{\cos^2 \theta - 1}}{\sin \theta} = \frac{\cos \theta \pm \sqrt{-\sin^2 \theta}}{\sin \theta}$$

$$= \frac{\cos \theta \pm i \sin \theta}{\sin \theta} = (\cos \theta \pm i \sin \theta) \operatorname{cosec} \theta$$

Let  $\alpha = (\cos \theta + i \sin \theta) \operatorname{cosec} \theta$  ,  $\beta = (\cos \theta - i \sin \theta) \operatorname{cosec} \theta$  .

$$\therefore \alpha^n = (\cos \theta + i \sin \theta)^n \operatorname{cosec}^n \theta = (\cos n\theta + i \sin(n\theta)) \operatorname{cosec}^n \theta$$

$$\beta^n = (\cos \theta - i \sin \theta)^n \operatorname{cosec}^n \theta = (\cos n\theta - i \sin(n\theta)) \operatorname{cosec}^n \theta$$

$$\therefore \alpha^n + \beta^n = 2 \cos n\theta \operatorname{cosec}^n \theta$$

$$\alpha^n \cdot \beta^n = (\cos \theta + i \sin \theta)^n \operatorname{cosec}^n \theta \cdot (\cos \theta - i \sin \theta)^n \operatorname{cosec}^n \theta$$

$$= (\cos^2 n\theta + \sin^2 n\theta) \operatorname{cosec}^{2n} \theta = \cos ec^{2n} \theta$$

**Q3)b) Solve the following equations by Gauss-Siedal Method ;**

**$15x+2y+z = 18$  ,  $2x+20y-3z = 19$  ,  $3x-6y+25z = 22$  .Take three iterations .** (6M)

**Ans :** We first write the equations as

$$x = \frac{1}{15} [18 - 2y - z] \quad \text{-----(1)}$$

$$y = \frac{1}{20} [19 - 2x + 3z] \quad \text{-----(2)}$$

$$z = \frac{1}{25} [22 - 3x + 6y] \quad \text{-----(3)}$$

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- (i) First Iteration : We start with the approximation  $y_0 = 0, z_0 = 0$  and then from (1) , we get

$$x_1 = \frac{18}{15} = 1.2 .$$

We use this approximation to find y from (2), i.e we put  $x=1.2$  and  $z=0$  in (2) and get

$$y_1 = \frac{1}{20} [19 - 2(1.2) + 3(0)] = \frac{16.6}{20} = 0.83 .$$

We use these values of x and y to find z from (3) i.e. we put  $x=1.2$  ,  $y=0.83$  in (3) and get

$$z = \frac{1}{25} [22 - 3(1.2) + 6(0.83)] = \frac{23.38}{25} = 0.9352 .$$

- (ii) Second iteration : We use the latest values of y and z in (1) to find x , i.e we put  $y=0.83$  and  $z=0.9352$  in (1) and get

$$x_2 = \frac{1}{15} [18 - 2(0.83) - 0.9352] = \frac{15.4048}{15} = 1.0270 .$$

We now put  $x=1.027$  and  $z=0.9352$  in (2) and get

$$y_2 = \frac{1}{20} [19 - 2(1.027) + 3(0.9352)] = \frac{19.7516}{20} = 0.9876 .$$

We use these values of x and y to find z from (3)

$$z_2 = \frac{1}{25} [22 - 3(1.0270) + 6(0.9876)] = \frac{28.8446}{25} = 0.9938 .$$

- (iii) Third Iteration : We use the latest values of y and z to find x , i.e. ,we put 0.9876 and  $z=0.9938$  in (1) and get

$$x_3 = \frac{1}{15} [18 - 2(0.9876) - 0.9938] = \frac{15.031}{15} = 1.0021 .$$

We now put  $x=1.0021$  and  $z=0.9938$  in (2) and get

$$y_3 = \frac{1}{20} [19 - 2(1.0021) + 3(0.9938)] = \frac{19.9772}{20} = 0.9989 .$$

We use these values of x and y to find z from (3)

$$z_3 = \frac{1}{25} [22 - 3(1.0021) + 6(0.9989)] = \frac{24.9871}{25} = 0.9995 .$$

Hence , we get  $x = 1.0021$  ,  $y = 0.9989$  ,  $z = 0.9995$  .

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**Q3)c) Prove that if z is a homogeneous function of two variables x and y of degree n , then**

$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z$  . Hence find the value of  $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2}$  at

**x=1 , y=1 when**  $z = x^6 \tan^{-1} \left( \frac{x^2 + y^2}{x^2 + xy} \right) + \frac{x^4 + y^4}{x^2 + y^2}$  . **(8M)**

**Ans :** Since z is a homogeneous function of degree n in x and y , by Euler's Theorem ,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \quad \text{----- (1)}$$

Differentiating (1) partially w.r.t x ,

$$\left( x \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} \cdot 1 \right) + y \frac{\partial^2 z}{\partial x \partial y} = n \frac{\partial z}{\partial x}$$

$$x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} = (n-1) \frac{\partial z}{\partial x}$$

Differentiating (1) partially w.r.t y ,

$$x \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} + y \frac{\partial^2 z}{\partial y^2} = n \frac{\partial z}{\partial y}$$

$$x \frac{\partial^2 z}{\partial x \partial y} + y \frac{\partial^2 z}{\partial y^2} = (n-1) \frac{\partial z}{\partial y}$$

Multiplying (2) by x and (3) by y and adding , we get ,

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = (n-1) \left[ x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right] = (n-1)nz$$

Further, if u is a homogeneous function of three variables x , y , z of degree n then we can prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + z^2 \frac{\partial^2 u}{\partial z^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + 2xy \frac{\partial^2 u}{\partial y \partial z} + 2xy \frac{\partial^2 u}{\partial z \partial x} = n(n-1)u$$

For  $z = x^6 \tan^{-1} \left( \frac{x^2 + y^2}{x^2 + xy} \right) + \frac{x^4 + y^4}{x^2 + y^2}$  .

Putting X=xt , Y=yt , we get

$$F(X,Y) = X^6 \tan^{-1} \left( \frac{X^2 + Y^2}{X^2 + XY} \right) + \frac{X^4 + Y^4}{X^2 + Y^2}$$



$$\therefore f(X, Y) = x^6 t^6 \tan^{-1} \left( \frac{x^2 t^2 + y^2 t^2}{x^2 t^2 + xtyt} \right) + \frac{x^4 t^4 + y^4 t^4}{x^2 t^2 + y^2 t^2} = x^6 t^6 \tan^{-1} \left( \frac{x^2 + y^2}{x^2 + xy} \right) + t^2 \frac{x^4 + y^4}{x^2 + y^2}$$

Now, let  $x^6 t^6 \tan^{-1} \left( \frac{x^2 + y^2}{x^2 + xy} \right) = u$ , and  $t^2 \frac{x^4 + y^4}{x^2 + y^2} = v$ .

u and v are homogeneous functions of degree 6 and 2 respectively .

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 6u, \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (n-1)nu = 30u$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 2v$$

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = (n-1)nv = 2v$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = (n-1)nu + n(n-1)v = 30u + 2v .$$

$$\therefore x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = (n-1)nz = 30x^6 t^6 \tan^{-1} \left( \frac{x^2 + y^2}{x^2 + xy} \right) + 2t^2 \frac{x^4 + y^4}{x^2 + y^2}$$

**Q4)a) If  $\tan(\alpha + i\beta) = \cos \theta + i \sin \theta$  then prove that  $\alpha = \frac{n\pi}{2} + \frac{\pi}{4}, \beta = \frac{1}{2} \log \left( \frac{\pi}{4} + \frac{\theta}{2} \right)$ . (6M)**

**Ans :** We have ,  $\tan(\alpha + i\beta) = \cos \theta + i \sin \theta \quad \therefore \tan(\alpha - i\beta) = \cos \theta - i \sin \theta$

$$\begin{aligned} \therefore \tan 2\alpha &= [\tan((\alpha + i\beta) + (\alpha - i\beta))] \\ &= \frac{\tan(\alpha + i\beta) + \tan(\alpha - i\beta)}{1 - \tan(\alpha + i\beta)\tan(\alpha - i\beta)} = \frac{2 \cos \theta}{1 - (\cos^2 \theta + \sin^2 \theta)} \end{aligned}$$

$$\tan 2\alpha = \frac{2 \cos \theta}{0}$$

$$\therefore 2\alpha = n\pi + \frac{\pi}{2}, \alpha = \frac{n\pi}{2} + \frac{\pi}{4}$$

Also ,  $\therefore \tan 2\beta = [\tan((\alpha + i\beta) - (\alpha - i\beta))] .$

$$= \frac{\tan(\alpha + i\beta) - \tan(\alpha - i\beta)}{1 + \tan(\alpha + i\beta)\tan(\alpha - i\beta)} = \frac{2i \sin \theta}{1+1} = i \sin \theta$$

$$i \tanh 2\beta = i \sin \theta$$

$$\therefore \tanh 2\beta = \sin \theta$$

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$$\therefore 2\beta = \tanh^{-1}(\sin \theta) = \frac{1}{2} \log \left( \frac{1 + \sin \theta}{1 - \sin \theta} \right)$$

$$\text{But } 1 + \sin \theta = \left( \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} \right) + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \left( \sin \frac{\theta}{2} + \cos \frac{\theta}{2} \right)^2$$

$$\text{But } 1 - \sin \theta = \left( \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} \right) - 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \left( \sin \frac{\theta}{2} - \cos \frac{\theta}{2} \right)^2$$

$$\therefore 2\beta = \frac{1}{2} \log \left( \frac{\left( \sin \frac{\theta}{2} + \cos \frac{\theta}{2} \right)^2}{\left( \sin \frac{\theta}{2} - \cos \frac{\theta}{2} \right)^2} \right) = \log \left[ \frac{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}} \right]$$

$$\beta = \frac{1}{2} \log \left[ \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}} \right] = \frac{1}{2} \log \tan \left( \frac{\pi}{4} + \frac{\theta}{2} \right)$$

**Q4)b) Expand  $x^5 + x^3 - x^2 + x - 1$  in powers of  $(x-1)$  and hence find the value of (6M)**

(1)  $f\left(\frac{9}{10}\right)$

(2)  $f(1.01)$

**Ans :** Let  $f(x) = x^5 + x^4 - x^2 + x - 1$  and  $a=1$ ,  $\therefore f(1) = 1$

$$\therefore f'(x) = 5x^4 - 4x^3 + 3x^2 - 2x + 1, \quad \therefore f'(1) = 3$$

$$\therefore f''(x) = 20x^3 - 12x^2 + 6x - 2, \quad \therefore f''(1) = 12$$

$$\therefore f'''(x) = 60x^2 - 24x + 6, \quad \therefore f'''(1) = 42$$

$$\therefore f^{(4)}(x) = 120x - 24, \quad \therefore f^{(4)}(1) = 96$$

$$\therefore f^{(5)}(x) = 120, \quad \therefore f^{(5)}(1) = 120$$

Now,  $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$

$$\therefore f(x) = 1 + (x-1).3 + \frac{(x-1)^2}{2!}.12 + \frac{(x-1)^3}{3!}.42 + \frac{(x-1)^4}{4!}.96 + \frac{(x-1)^5}{5!}.120 + \dots$$

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$$\therefore f(x) = 1 + (x-1).3 + 6(x-1)^2 + 7(x-1)^3 + 4(x-1)^4 + (x-1)^5 \quad .$$

(i) To find  $f\left(\frac{9}{10}\right)$ , we put  $x=0.9$ , and  $x-1 = -0.1$

$$\begin{aligned} \therefore f(0.9) &= 1 + 3(-0.1) + 6(-0.1)^2 + 7(-0.1)^3 + 4(-0.1)^4 + (-0.1)^5 \\ &= 1 - 0.3 + 0.06 - 0.007 + 0.0004 - 0.00005 \\ &= 0.7534 \end{aligned} \quad .$$

(ii) To find  $f(1.01)$ , we put  $x=1.01$  and  $(x-1)=0.01$  .

$$\begin{aligned} \therefore f(1.01) &= 1 + 3(0.01) + 6(0.01)^2 + 7(0.01)^3 + 4(0.01)^4 + (0.01)^5 \\ &= 1 + 0.03 + 0.0006 - 0.000007 + 0.00000004 - 0.0000000005 \\ &= 01.0306 \end{aligned} \quad .$$

**Q4)c) For what values of  $\lambda$  and  $\mu$ , the equations,  $x + y + z = 6$ ;  $x + 2y + 3z = 10$ ;  $x + 2y + \lambda z = \mu$ ;**

- (i) Have a unique solution
- (ii) Have infinite solution

**Find the solution in each case for a possible value of  $\mu$  and  $\lambda$  .**

**(8M)**

**Ans :**

We have 
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} \quad .$$

By  $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$  ,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu - 10 \end{bmatrix} \quad .$$

(i) The system has unique solution if the coefficient matrix is non-singular (or the rank A ,  
r = the number of unknowns , n=3 )

This requires  $\lambda - 3 \neq 0$ ,  $\therefore \lambda \neq 3$  .

$\therefore \lambda \neq 3$  then ( $\mu$  may have any value )the system has unique solution .

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(ii) If  $\lambda = 3$ , the coefficient matrix and the augmented matrix becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & \mu - 10 \end{bmatrix}$$

The rank of  $A = 2$ , and the rank of  $[A, B]$  will be also 2 if  $\mu = 10$ .

Thus if  $\lambda = 3$  and  $\mu = 10$ , the system is consistent. But the rank of  $A (=2)$  is less than the number of unknowns ( $=3$ ). Hence the equations will possess infinite solutions.

**Q5)a) Find the nth derivative of  $y = \frac{1}{x^2 + a^2}$ . (6M)**

**Ans :** We have

$$\begin{aligned} y &= \frac{1}{x^2 + a^2} = \frac{1}{x^2 - a^2 i^2} = \frac{1}{2ai} \left[ \frac{1}{x - ai} - \frac{1}{x + ai} \right] \\ y_n &= \frac{1}{2ai} \left[ \frac{(-1)^n \cdot n!}{(x - ai)^{n+1}} - \frac{(-1)^n \cdot n!}{(x + ai)^{n+1}} \right] \\ &= \frac{(-1)^n \cdot n!}{2ai} \left[ \frac{1}{(x - ai)^{n+1}} - \frac{1}{(x + ai)^{n+1}} \right] \end{aligned}$$

Let  $x = r \cos \theta, a = r \sin \theta$ , so that  $r^2 = x^2 + a^2, \theta = \tan^{-1}(a/x)$ .

Now,

$$\begin{aligned} \frac{1}{(x - ai)^{n+1}} &= \frac{1}{r^{n+1} (\cos \theta - i \sin \theta)^{n+1}} = \frac{1}{r^{n+1}} \cdot \frac{1}{\cos(n+1)\theta - i \sin(n+1)\theta} \\ \frac{1}{(x - ai)^{n+1}} &= \frac{1}{r^{n+1}} [\cos(n+1)\theta + i \sin(n+1)\theta] \\ \frac{1}{(x + ai)^{n+1}} &= \frac{1}{r^{n+1} (\cos \theta + i \sin \theta)^{n+1}} = \frac{1}{r^{n+1}} \cdot \frac{1}{\cos(n+1)\theta + i \sin(n+1)\theta} \\ \frac{1}{(x + ai)^{n+1}} &= \frac{1}{r^{n+1}} [\cos(n+1)\theta - i \sin(n+1)\theta] \\ \therefore \frac{1}{(x - ai)^{n+1}} - \frac{1}{(x + ai)^{n+1}} &= \frac{1}{r^{n+1}} \cdot 2i \sin(n+1)\theta \end{aligned}$$

Putting these values in  $\frac{(-1)^n \cdot n!}{2ai} \left[ \frac{1}{(x - ai)^{n+1}} - \frac{1}{(x + ai)^{n+1}} \right]$ , we get

$$y_n = (-1)^n \cdot n! \cdot \frac{1}{a} \cdot \frac{1}{r^{n+1}} \sin(n+1)\theta$$

$$\text{But } r = \frac{a}{\sin \theta} \quad \left( \because a = r \sin \theta, \therefore r^{n+1} = \frac{a^{n+1}}{\sin^{n+1} \theta} \right)$$

$$y_n = (-1)^n \cdot n! \cdot \frac{1}{a^{n+2}} \sin^{n+1} \theta \sin(n+1)\theta$$

**Q5)b) Discuss the maxima and minima of  $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 16$  . (6M)**

**Ans :** We have  $f(x, y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 16$  .

Step I :

$$f_x = 3x^2 + y^2 - 24x + 21$$

$$f_y = 2xy - 4y$$

$$f_{xx} = 6x - 24, f_{xy} = 2y, f_{yy} = 2x - 4$$

Step II :

We now solve the equations  $f_x = 0$  ,  $f_y = 0$  .

$$\therefore 3x^2 + y^2 - 24x + 21 = 0 \text{ and } 2xy - 4y = 0$$

The second equation gives  $2y(x-2) = 0$  .

$$\therefore x = 2 \text{ or } y = 0$$

When  $x=2$  , the first equation  $3x^2 + y^2 - 24x + 21 = 0$  gives

$$\therefore 12 + y^2 - 48 + 21 = 0 \text{ hence , } y^2 - 15 = 0 \text{ , } y^2 = 15 \text{ , } y = \pm\sqrt{15}$$

$$\therefore \text{The stationary values are } (2, \sqrt{15}) \text{ , } (2, -\sqrt{15})$$

When  $y = 0$  , the first equation  $3x^2 + y^2 - 24x + 21 = 0$  gives

$$3x^2 - 24x + 21 = 0 \text{ , } x^2 - 8x + 7 = 0$$

$$(x-7)(x-1) = 0 \text{ , hence } x = 1, 7$$

Therefore , the stationary values are  $(1,0)$  ,  $(7,0)$  .

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Step III :

(i) For  $x=2, y=\sqrt{15}$  ,

$$r = f_{xx} = 12 - 24 = -12, s = f_{xy} = 2\sqrt{15}, t = f_{yy} = 4 - 4 = 0$$

$$\therefore rt - s^2 = 0 - 60 = -60 < 0$$

$\therefore f(x,y)$  is neither maximum nor minimum. It is a saddle point .

(ii) For  $x=2, y=-\sqrt{15}$  ,

$$r = f_{xx} = 12 - 24 = -12, s = f_{xy} = -2\sqrt{15}, t = f_{yy} = 4 - 4 = 0$$

$$\therefore rt - s^2 = 0 - 60 = -60 < 0$$

$\therefore f(x,y)$  is neither maximum nor minimum. It is a saddle point .

(iii) For  $x=1, y=0$  ,

$$r = f_{xx} = 6 - 24 = -18, s = f_{xy} = 0, t = f_{yy} = 2 - 4 = -2$$

$$\therefore rt - s^2 = 36 - 0 = 36 > 0 \text{ and } r = -18, \text{ negative}$$

$\therefore (1,0)$  is a maxima .

$\therefore$  The maximum value =  $1 + 0 - 12 - 0 + 21 = 20$  .

(iv) For  $x=7, y=0$  ,

$$r = f_{xx} = 42 - 24 = 18, s = f_{xy} = 0, t = f_{yy} = 14 - 4 = 10$$

$$\therefore rt - s^2 = 180 - 0 = 180 > 0$$

Hence ,  $(7,0)$  is a minima .

The minimum value =  $343 + 0 - 588 - 0 + 147 + 10 = -88$  .

**Q5)c) Prove that if A and B are two unitary matrices then AB is also unitary. Verify the result when**

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} . \quad (8M)$$

**Ans :** We have

$$(AB)(AB)^\theta = (AB)(B^\theta A^\theta) = A(BB^\theta)A^\theta$$

$$= AIA^\theta \quad [ \because B \text{ is unitary } ]$$

$$= AA^\theta = I \quad [ \because A \text{ is unitary } ]$$

Similarly , we can prove that  $(AB)^\theta(AB) = I$  .

Hence , AB is also unitary .

$$\text{Now, } A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} .$$

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$$\therefore A' = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$A^\theta = (\bar{A}') = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\therefore A' = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$A^\theta = (\bar{A}') = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^\theta A' = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^\theta A' = \frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Now,  $B = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$

$$\therefore B' = \frac{1}{2} \begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix}, \quad \therefore B^\theta = (\bar{B}') = \frac{1}{2} \begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$B^\theta B = \frac{1}{4} \begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix} \begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

$$B^\theta B = \frac{1}{4} \begin{bmatrix} (1-i)^2 + (1-i)^2 & -(1-i)^2 + (1-i)^2 \\ -(1+i)^2 + (1+i)^2 & (1-i)^2 + (1-i)^2 \end{bmatrix}$$

$$B^\theta B = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence it is proved that if A and B are two unitary matrices then AB is also unitary and the result is verified .

**Q6)a) If  $x = \cosh\left(\frac{1}{m} \log y\right)$ , prove that  $(x^2 - 1)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0$ . (6M)**

**Ans :**  $x = \cosh\left(\frac{1}{m} \log y\right)$  .

$$\cosh^{-1} x = \left(\frac{1}{m} \log y\right)$$

$$\log y = m \log\left(x + \sqrt{x^2 - 1}\right) = \log\left(x + \sqrt{x^2 - 1}\right)^m$$

Differentiating w.r.t  $x$ ,

$$y_1 = m\left(x + \sqrt{x^2 - 1}\right)^{m-1} \left(1 + \frac{2x}{2\sqrt{x^2 - 1}}\right)$$

$$y_1 = m\left(x + \sqrt{x^2 - 1}\right)^{m-1} \left(\frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}}\right)$$

$$= m \frac{\left(\sqrt{x^2 - 1} + x\right)^m}{\sqrt{x^2 - 1}} = \frac{my}{\sqrt{x^2 - 1}}$$

$$y_1 \sqrt{x^2 - 1} = my$$

$$(x^2 - 1)y = m^2 y^2$$

Differentiating w.r.t  $x$ ,

$$y_1 \sqrt{x^2 - 1} = my$$

$$(x^2 - 1)2y_1 y_2 + 2xy_1^2 = m^2 2yy_1$$

$$(x^2 - 1)y_2 + xy_1 = m^2 y$$

Differentiating n times using Leibnitz Theorem ,

$$(x^2 - 1)y_{n+2} + n.2xy_{n+1} + \frac{n(n-1)}{2!} 2y_n + xy_{n+1} + ny_n = m^2 y_n$$

$$(x^2 - 1)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0$$

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**Q6)b) Find a root of the equation  $xe^x = \cos x$  using the Regula Falsi Method correct to three decimal places . (6M)**

**Ans :**

$$f(x) = \cos x - xe^x = 0$$

$$f(0) = 1$$

$$f(1) = \cos 1 - e = -2.17798$$

The root lies between 0 and 1 .

Taking  $x_0 = 0, x_1 = 1, f(x_0) = 1, f(x_1) = -2.17798$  ,

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{-1}{-2.17798 - 1} = \frac{1}{3.17798} = 0.3147$$

Now,  $\cos 0.3147 - 0.3147e^{0.3147} = 0.5199$  .

The value that we get is positive , so the root lies between 0.3147 and 1.

Taking  $x_0 = 0.3147, x_1 = 1, f(x_0) = 0.5199, f(x_1) = -2.17798$  ,

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0.3147(2.1779) + 0.685335}{-2.17798 - (0.51987)} = \frac{1}{2.2384} = 0.44675$$

Now,

$$\cos 0.44675 - 0.44675e^{0.44675} = 0.2035$$

The value that we get is positive , so the root lies between 0.44675 and 1 .

Taking  $x_0 = 0.44675, x_1 = 1, f(x_0) = 0.2035, f(x_1) = -2.17798$  ,

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0.44675(-2.1779) - 1 \times 0.2035}{-2.17798 - (0.2035)} = \frac{1}{2.0242} = 0.494020$$

Now,

$$\cos 0.494020 - 0.494020e^{0.494020} = 0.0708$$

The value that we get is positive , so the root lies between 0.494020 and 1 .

Taking  $x_0 = 0.494020, x_1 = 1, f(x_0) = 0.0708, f(x_1) = -2.17798$

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Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0.494020(-2.1779) - 1 \times 0.0708}{-2.17798 - (0.0708)} = \frac{1}{1.9316} = 0.51771 \text{ .}$$

Now,

$$\cos 0.51771 - 0.51771 e^{0.51771} = 0.00124 \text{ .}$$

Taking ,  $x_0 = 0.51771, x_1 = 1, f(x_0) = 0.00124, f(x_1) = -2.17798$  .

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0.51771(-2.1779) - 1 \times 0.00124}{-2.17798 - (0.0708)} = \frac{1}{1.9315699} = 0.5177136 \text{ .}$$

$$\text{Now, } \cos 0.5177136 - 0.5177136 e^{0.5177136} = 0.00124 \text{ .}$$

If we compare  $x_2$  and  $x_0$  , we find that both are same upto four decimal places .

Hence , the root of the equation correct upto four decimal places is 0.5177 .

**Q6)c)1) Expand  $\sin^4 \theta \cos^2 \theta$  in a series of multiples of  $\theta$  .**

**(4M)**

**Ans :** Let  $x = \cos \theta + i \sin \theta$   $\therefore \frac{1}{x} = \cos \theta - i \sin \theta$

Also  $x^n + \frac{1}{x^n} = 2 \cos n\theta$  and  $x^n - \frac{1}{x^n} = 2i \sin n\theta$  .

Now consider ,

$$\begin{aligned} (2i \sin \theta)^4 (2 \cos \theta)^3 &= \left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^3 \\ &= \left(x - \frac{1}{x}\right)^3 \left(x - \frac{1}{x}\right) \left(x + \frac{1}{x}\right)^3 = \left(x^2 - \frac{1}{x^2}\right)^3 \left(x - \frac{1}{x}\right)^3 \\ &= \left(x^6 - 3x^2 + 3 \cdot \frac{1}{x^2} - \frac{1}{x^6}\right) \left(x - \frac{1}{x}\right) \\ &= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} - x^5 + 3x - \frac{3}{x^3} + \frac{1}{x^7} \\ &= \left(x^7 + \frac{1}{x^7}\right) - \left(x^5 + \frac{1}{x^5}\right) - 3\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right) \end{aligned}$$

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$$(2i \sin \theta)^4 (2 \cos \theta)^3 = 2 \cos 7\theta - 2 \cos 5\theta - 6 \cos 3\theta + 6 \cos \theta \quad .$$

$$\sin^4 \theta \cos^3 \theta = \frac{\cos 7\theta}{2^6} - \frac{\cos 5\theta}{2^6} - \frac{3 \cos 3\theta}{2^6} + \frac{3 \cos \theta}{2^6} \quad .$$

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**Q6)c)2) If one root of  $x^4 - 6x^3 + 18x^2 - 24x + 16 = 0$  is  $(1+i)$  ; find the other roots. (4M)**

**Ans :** Since  $(1+i)$  is a root of the given equation , then we know that  $(1-i)$  must be one of the remaining roots because complex roots always occur in conjugate pairs . Hence ,  $(x-1-i)$  and  $(x-1+i)$  are the factors of the left hand side , i.e the left hand side is divisible by

$$\{(x-1-i)\} \cdot \{(x-1+i)\} , \text{ i.e. by } (x-1)^2 - i^2 = x^2 - 2x + 2 \quad .$$

Dividing the left hand side by  $x^2 - 2x + 2$  , we get  $x^2 - 8x + 32$  .

Solving the equation  $x^2 - 8x + 32$  , we get  $x^2 = 4 \pm 4i$  .

Hence , the remaining roots are  $(1-i), (4+4i), (4-4i)$  .

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